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INFINITE PARTITION REGULAR MATRICES

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We consider infinite matrices with entries from $\mathbb{Z}$ (and only finitely many nonzero entries on any row). A matrix A is partition regular over $\mathbb{N}$ provided that, whenever the set $\mathbb{N}$ of positive integers is partitioned into finitely many classes there is a vector $\vec{x}$ with entries in $\mathbb{Z}$ such that all entries of $A\vec{x}$ lie in the same cell of the partition. We show that, in marked contrast with the situation for finite matrices, there exists a finite partition of $\mathbb{N}$ no cell of which contains solutions for all partition regular matrices and determine which of our pairs of matrices must always have solutions in the same cell of a partition.

1. Introduction

Let A be a matrix with integer entries. If A has infinitely many columns we demand also that each row of A has only finitely many nonzero entries. We say that A is partition regular (over $\mathbb{N}$) provided that, whenever the set $\mathbb{N}$ of positive integers is partitioned into finitely many classes there is a vector $\vec{x}$ with entries in $\mathbb{Z}$ (and with the same number of entries as A has columns) such that all entries of $A\vec{x}$ lie in the same cell of the partition. In this case we say that the cell of the partition contains a "solution" to A.

The situation in the event A is finite is well understood. The classification of partition regular matrices is in terms of "$(m,p,c)-$sets" from [2]. We should remark that this is an alternative definition of a partition regular matrix. Instead of asking that the entries of $A\vec{x}$ lie in one cell of a partition, one asks that one can get $\vec{x}$ with all entries in the same cell such that $A\vec{x}=\vec{0}$ (kernel partition regular). In the case of finite matrices, the two definitions give rise to equivalent theories.

Definition 1.1. Let m,p,c be in $\mathbb{N}$.

(a) Let $\vec{x}\in\mathbb{N}^m$. $S(m,p,c,\vec{x})=\Big\{cx_t+\sum\limits_{i=t+1}^{m}\lambda_i x_i : t\in\{1,2,\dots,m\}$ and for $t<i\le m, \lambda_i\in\mathbb{Z}$ with $|\lambda_i|\le p\Big\}$.

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(b) A subset $B \subseteq \mathbb{N}$ is an (m,p,c)-set if and only if there is some $\vec{x} \in \mathbb{N}^m$ such that $B = S(m,p,c,\vec{x})$.

(c) An $l \times m$ matrix A is an (m,p,c)-matrix if and only if whenever $\vec{x} \in \mathbb{N}^m$, and $A\vec{x} = \vec{y}$ one has $S(m,p,c,\vec{x}) = \{y_1, y_2, \ldots, y_\ell\}$ (where ℓ is the number of rows of A).

For example, the matrix

$$\begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{is a (2,2,1)-matrix.}$$

The key to the characterization of partition regular finite matrices is the following theorem.

Theorem 1.2. *Let A be a $u \times v$ matrix with integer entries. Then A is partition regular if and only if there exist m,p,c in $\mathbb{N}$ and an (m,p,c)-matrix B so that whenever $\vec{y} \in \mathbb{Z}^m$ there exists $\vec{x} \in \mathbb{Z}^v$ with all entries of $A\vec{x}$ included in the entries of $B\vec{y}$.*

Proof. We only sketch the proof. For a related argument we refer to [6].

By [2], for every $(m,p,c) \in \mathbb{N}^3$ and for every partition of $\mathbb{N}$ into finitely many classes, some one of these classes contains an (m,p,c)-set. This establishes the "only if" part.

For the "if" part we use the following result from [2]: A matrix A is kernel partition regular, if there exist $(m,p,c) \in \mathbb{N}^3$ such that every (m,p,c)-set contains a solution of $A\vec{x} = \vec{0}$. Now, let $A\vec{x} = \vec{y}$ be partition regular. By substituting successively the x_i's, we obtain a system $A'\vec{y} = \vec{0}$ (which might be the equation $0=0$). Now applying the above mentioned results finishes the proof. ∎

Definition 1.3. *A* subset *D* of $\mathbb{N}$ is "large" if and only if for all m,p,c in $\mathbb{N}$ there exists $\vec{x} \in \mathbb{N}^m$ with $S(m,p,c,\vec{x}) \subseteq D$.

Thus a large subset of $\mathbb{N}$ contains solutions for all partition regular (finite) matrices by Theorem 1.2. The fundamental result to complete the understanding of finite partition regular matrices is the following.

Theorem 1.4. *If $\mathbb{N}$ is partitioned into finitely many cells, one of these cells is large.*

Proof. This follows from [2, Satz 3.1]. ∎

One can define a corresponding notion of large for infinite partition regular matrices and ask whether the analogue of Theorem 1.4 remains true. We shall obtain in Section 3 a strong negative answer to this question.

Earlier investigations of infinite partition regular matrices led to the notion of an (m,p,c)-system [3], [7]. The notation below generalizes

$$FS(\langle x_n \rangle_{n=1}^{\infty}) = \Big\{ \sum_{n \in F} x_n : F \text{ is finite nonempty subset of } \mathbb{N} \Big\}.$$

Definition 1.5. Fix (for the remainder of the paper) an enumeration

$$\langle (m(n), p(n), c(n)) \rangle_{n=1}^{\infty} \text{ of } \mathbb{N}^3.$$

(a) $V = \times_{n=1}^{\infty} \mathbb{N}^{m(n)}$.

(b) Given $\vec{x} \in V$, $S(\vec{x}, n) = S(m(n), p(n), c(n), \vec{x}(n)) =$

$$\Big\{ c(n) \cdot \vec{x}(n)(t) + \sum_{i=t+1}^{m(n)} \lambda_i \vec{x}(n)(i) : t \in \{1, 2, \ldots, m(n)\} \text{ and for }$$
$$i \in \{t+1, t+2, \ldots, m(n)\},\ \lambda_i \in \mathbb{Z} \text{ with } |\lambda_i| \le p(n) \Big\},$$

where $\vec{x}(n)(i)$ denotes the i'th entry of $\vec{x}(n)$.

(c) Given $\vec{x} \in V$ and $\ell \in \mathbb{N}$, $FS(\langle S(\vec{x}, n) \rangle_{n=\ell}^{\infty}) = \Big\{ \sum_{n \in F} w_n : F$ is a finite non empty subset of $\mathbb{N}$ with $\min F \ge \ell$ and for each $n \in F, w_n \in S(\vec{x}, n) \Big\}$.

(d) A subset D of $\mathbb{N}$ is an (m, p, c)-system if and only if there exists some $\vec{x} \in V$ such that $D = FS(\langle S(\vec{x}, n) \rangle_{n=1}^{\infty})$.

In [3] it was shown that (m, p, c)-systems are partition regular in the sense that whenever $\mathbb{N}$ is partitioned into finitely many classes, some one of them contains an (m, p, c)-system. In [7] it was shown in fact that a stronger partition regularity holds: whenever an (m, p, c)-system is partitioned into finitely many classes some one of them contains another (m, p, c)-system.

We were led to conjecture that perhaps (m, p, c)-systems were universal for infinite partition regular matrices in the same sense (Theorem 1.2) that (m, p, c)-sets are universal for finite partition regular matrices. That is, given an infinite partition regular matrix A one would ask that for every $\vec{x} \in V$ there would exist some $\vec{y} \in \times_{n=1}^{\infty} \mathbb{Z}$ with all entries of $A\vec{y}$ in $FS(\langle S(\vec{x}, n) \rangle_{n=1}^{\infty})$. We shall introduce in Section 2 a class of infinite partition regular systems which provide a strong negative answer to this conjecture as well. We remark that the corresponding question for kernel partition regular matrices also turns out to have a negative answer — see Section 4.

In some of our proofs we utilize the topological-algebraic system $(\beta\mathbb{N}, +, \cdot)$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of the discrete space $\mathbb{N}$ and $+$ and $\cdot$ are the extensions of ordinary addition and multiplication to $\beta\mathbb{N}$ making $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$ left topological semigroups with $\mathbb{N}$ contained in their topological center. That is for each $q \in \beta\mathbb{N}$ the functions $\nu \to q + \nu$ and $\nu \to q \cdot \nu$ are continuous and for each $x \in \mathbb{N}$ the functions $\nu \to \nu + x$ and $\nu \to \nu \cdot x$ are continuous. See [5] for an elementary construction of the semigroups $(\beta\mathbb{N}, +)$ and $(\beta\mathbb{N}, \cdot)$, whose points we take to be ultrafilters on $\mathbb{N}$. We write $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$.

2. Milliken-Taylor systems

In 1975 K. Milliken and A. Taylor independently proved a generalization of Ramsey's Theorem which allows us to produce infinite partition regular matrices which do not arise as part of (m,p,c)-systems and which establish that two infinite partition regular matrices need not have solutions in the same cell of a partition.

Definition 2.1. Let $\langle D_n\rangle_{n=1}^{\infty}$ be a sequence of finite nonempty subsets of $\mathbb{N}_0$ and let $k\in\mathbb{N}$.

$[FU(\langle D_n\rangle_{n=1}^{\infty})]_<^k = \{(\cup_{n\in F_1}D_n,\cup_{n\in F_2}D_n,\ldots,\cup_{n\in F_k}D_n) : F_1,F_2,\ldots,F_k$ are finite nonempty subsets of $\mathbb{N}$ and for each $i\in\{1,2,\ldots,k-1\}$, $\max F_i<\min F_{i+1}\}$.

Theorem 2.2. (Milliken, Taylor) *Let $\langle D_n\rangle_{n=1}^{\infty}$ be a sequence of finite nonempty subsets of $\mathbb{N}_0$ such that for each $n\in\mathbb{N}$, $\max D_n<\min D_{n+1}$. Let $k,r\in\mathbb{N}$ and let $[FU(\langle D_n\rangle_{n=1}^{\infty})]_<^k=\cup_{i=1}^r B_i$. There exist $i\in\{1,2,\ldots,r\}$ and a sequence $\langle H_n\rangle_{n=1}^{\infty}$ of finite nonempty subsets of $\mathbb{N}$ such that for each n, $\max H_n<\min H_{n+1}$ and, if $E_n=\cup_{t\in H_n}D_t$, then $[FU(\langle E_n\rangle_{n=1}^{\infty})]_<^k\subseteq B_i$.*

Proof. [8, Theorem 2.2], [9, Lemma 2.2]. ∎

We introduce now systems motivated by this theorem.

Definition 2.3. Let $\vec{a}=(a_1,a_2,\ldots,a_k)$ be a finite sequence in $\mathbb{N}$ and let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. Then the Milliken-Taylor system for $\vec{a}$ is $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})=$ $\Big\{\sum_{t=1}^{k}a_t\cdot\sum_{n\in F_t}x_n : F_1,F_2,\ldots,F_k$ are finite nonempty subsets of $\mathbb{N}$ with $\max F_t<\min F_{t+1}$ for $t\in\{1,2,\ldots,k-1\}\Big\}$.

Observe that any Milliken-Taylor system $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})$ is generated by a matrix-in fact by any of uncountably many matrices obtained from one another by permuting rows. For example if

$$A=\begin{pmatrix}
1 & 2 & 0 & 0 & 0 & \ldots\\
0 & 1 & 2 & 0 & 0 & \ldots\\
1 & 0 & 2 & 0 & 0 & \ldots\\
1 & 1 & 2 & 0 & 0 & \ldots\\
1 & 2 & 2 & 0 & 0 & \ldots\\
0 & 0 & 1 & 2 & 0 & \ldots\\
0 & 1 & 0 & 2 & 0 & \ldots\\
0 & 1 & 1 & 2 & 0 & \ldots\\
0 & 1 & 2 & 2 & 0 & \ldots\\
1 & 0 & 0 & 2 & 0 & \ldots\\
1 & 0 & 1 & 2 & 0 & \ldots\\
1 & 0 & 2 & 2 & 0 & \ldots\\
1 & 1 & 0 & 2 & 0 & \ldots\\
1 & 1 & 1 & 2 & 0 & \ldots\\
1 & 1 & 2 & 2 & 0 & \ldots\\
1 & 2 & 0 & 2 & 0 & \ldots\\
1 & 2 & 2 & 2 & 0 & \ldots\\
 & & \vdots & & & \ldots
\end{pmatrix}$$

then $MT((1,2),\langle x_n\rangle_{n=1}^{\infty})$ is the set of entries in $A\vec{x}$.

Definition 2.4. If $\vec{a}$ is a finite sequence in $\mathbb{N}$ and A is an infinite matrix such that for each $\vec{x}=\langle x_n\rangle_{n=1}^{\infty}$, $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})$ is the set of entries of $A\vec{x}$, then A is said to generate Milliken-Taylor systems for $\vec{a}$.

Theorem 2.5. *Let $\vec{a}$ be a finite sequence in $\mathbb{N}$ and let $\langle y_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. Let $\mathbb{N}=\cup_{i=1}^{r}B_i$. Then there is a sequence $\langle H_n\rangle_{n=1}^{\infty}$ of finite nonempty subsets of $\mathbb{N}$ with $\max H_n<\min H_{n+1}$ for each $n\in\mathbb{N}$ and there is some $i\in\{1,2,\ldots,r\}$ such that, if $x_n=\sum_{\ell\in H_n} y_\ell$, then $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})\subseteq B_i$. In particular, if A is a matrix generating Milliken-Taylor systems for $\vec{a}$, then A is partition regular.*

Proof. For each n let $D_n=\{n\}$, so that $\cup_{n\in F}D_n=F$. Let k be the length of $\vec{a}$ and for $i\in\{1,2,\ldots,r\}$, let $\mathscr{B}_i=\{(F_1,F_2,\ldots,F_k)$: each F_t is a finite nonempty subset of $\mathbb{N}$ and for $t\in\{1,2,\ldots,k\}$, $\max F_t<\min F_{t+1}$ and $\sum_{t=1}^{k}a_t\cdot\sum_{n\in F_t}y_n\in B_i\}$. Pick $i\in\{1,2,\ldots,r\}$ and $\langle H_n\rangle_{n=1}^{\infty}$ as guaranteed by Theorem 2.2. For each n, let $x_n=\sum_{\ell\in H_n}y_\ell$.

To see that $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})\subseteq B_i$, let $F_1,F_2,\ldots,F_k$ be finite nonempty subsets of $\mathbb{N}$ with $\max F_t<\min F_{t+1}$, for $t\in\{1,2,\ldots,k-1\}$. For each n, $H_n=\cup_{\ell\in H_n}D_\ell$ so $[FU(\langle H_n\rangle_{n=1}^{\infty})]_<^k\subseteq\mathscr{B}_i$. For each $t\in\{1,2,\ldots,k\}$, let $G_t=\cup_{n\in F_t}H_n$. Then $(G_1,G_2,\ldots,G_k)\in[FU(\langle H_n\rangle_{n=1}^{\infty})]_<^k$ so $\sum_{t=1}^{k}a_t\cdot\sum_{\ell\in G_t}y_\ell\in B_i$. Since $\sum_{t=1}^{k}a_t\cdot\sum_{\ell\in G_t}y_\ell=\sum_{t=1}^{k}a_t\cdot\sum_{n\in F_t}\sum_{\ell\in H_n}y_\ell=\sum_{t=1}^{k}a_t\cdot\sum_{n\in F_t}x_n$, we are done. ∎

As an immediate consequence, we see that Milliken-Taylor systems are themselves partition regular.

Corollary 2.6. *Let $\langle y_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$, let $\vec{a}$ be a finite sequence in $\mathbb{N}$, and let $MT(\vec{a},\langle y_n\rangle_{n=1}^{\infty})=\cup_{i=1}^{r}B_i$. Then there exist $i\in\{1,2,\ldots,r\}$ and a sequence $\langle x_n\rangle_{n=1}^{\infty}$, such that $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})\subseteq B_i$.* ∎

The Milliken-Taylor systems provide already a counterexample to our conjecture about universality of (m,p,c)-systems. Although it is not hard to verify this directly we omit the verification here since the conclusion is an immediate corollary of our results in Section 3.

We now extend the Milliken-Taylor systems in the same way that (m,p,c)-systems extend finite sums.

Definition 2.7. Let $\vec{x}\in V$ and let $\vec{a}=(a_1,a_2,\ldots,a_k)$ be a finite sequence in $\mathbb{N}$ and let $\ell\in\mathbb{N}$. Then $MT(\vec{a},\langle S(\vec{x},n)\rangle_{n=\ell}^{\infty})=\Big\{\sum_{t=1}^{k}a_t\cdot\sum_{n\in F_t}w_n : F_1,F_2,\ldots,F_k$ are finite nonempty subsets of $\mathbb{N}$ with $\ell \leq \min F_1$ and with $\max F_t < \min F_{t+1}$ for $t \in \{1,2,\ldots,k-1\}$ and if $n \in \cup_{t=1}^{k} F_t$, then $w_n \in S(\vec{x},n)\Big\}$. We call such a system a Milliken-Taylor (m,p,c)-system.

Then $MT((1),\langle S(\vec{x},n)\rangle_{n=1}^{\infty}) = FS(\langle S(\vec{x},n)\rangle_{n=1}^{\infty})$. That is, the Milliken-Taylor (m,p,c) systems generalize the ordinary (m,p,c)-systems. As with the ordinary Milliken-Taylor systems, the Milliken-Taylor (m,p,c)-systems are generated by matrices.

In order to state our theorem about the partition regularity of Milliken-Taylor (m,p,c)-systems in their full generality, we need to introduce some more notation from [7]. Recall that we are taking the points of $\beta\mathbb{N}$ to be ultrafilters on $\mathbb{N}$ (the points of $\mathbb{N}$ being identified with the principal ultrafilters). For $q \in \beta\mathbb{N}$ and $A \subseteq \mathbb{N}$, the statements "$A \in q$" and "$cl_{\beta\mathbb{N}} A$ is a neighborhood of q" are synonymous.

Definition 2.8.

(a) $U = \{q \in \beta\mathbb{N}$: for each $A \in q$ there exists $\vec{x} \in V$ such that for all $n \in \mathbb{N}$, $S(\vec{x},n) \subseteq A\}$.

(b) Given $\vec{x} \in V, T(\vec{x}) = U \cup \bigcap_{l=1}^{\infty} cl_{\beta\mathbb{N}} FS(\langle S(\vec{x},n)\rangle_{n=\ell}^{\infty})$.

(c) $V^* = \{\vec{x} \in V$: for each $n \in \mathbb{N}$, each $a \in S(\vec{x},n)$, and each $b \in S(\vec{x},n+1)$, if $t = \max\{i : 2^i \leq a\}$, then 2^{t+1} divides $b\}$.

(d) For $\vec{x} \in V^*$ and $a \in FS(\langle S(\vec{x},n)\rangle_{n=1}^{\infty}), F(\vec{x},a)$ is the unique finite nonempty subset of $\mathbb{N}$ such that there is a choice of $w_n \in S(\vec{x},n)$ for each $n \in F(\vec{x},a)$ so that $a = \sum_{n \in F(\vec{x},a)} w_n$.

(e) For $\vec{x} \in V^*$ we say $\vec{y}$ refines $\vec{x}$ if and only if $\vec{y} \in V^*, FS(\langle S(\vec{y},n)\rangle_{n=1}^{\infty}) \subseteq FS(\langle S(\vec{x},n)\rangle_{n=1}^{\infty})$ and for each $n \in \mathbb{N}$, each $a \in S(\vec{y},n)$, and each $b \in S(\vec{y},n+1)$ one has $\max F(\vec{x},a) < \min F(\vec{x},b)$.

Observe that if $\vec{x}$, $\vec{y}$, and $\vec{z}$ are in V^*, $\vec{y}$ refines $\vec{x}$ and $\vec{z}$ refines $\vec{y}$, then $\vec{z}$ refines $\vec{x}$.

One can achieve the "in particular" part of the following Theorem 2.10 in a fashion similar to the derivation of Theorem 2.5, replacing the appeal to the Milliken-Taylor Theorem (Theorem 2.2) by an appeal to [7]. However, Theorem 2.10 gives additional information about the algebraic structure of $\beta\mathbb{N}$. Observe that for $a \in \mathbb{N}$ and p in $\beta\mathbb{N}$, by $a \cdot p$ we mean the product of a and p in $(\beta\mathbb{N},\cdot)$ and not the sum of p with itself a times. In particular, given $A \subseteq \mathbb{N}, A \in a \cdot p$ if and only if $A/a \in p$ where $A/a = \{x \in \mathbb{N} : x \cdot a \in A\}$ (and of course given $p,q \in \beta\mathbb{N}$ and $A \subseteq \mathbb{N}, A \in p+q$ if and only if $\{x \in \mathbb{N} : A - x \in p\} \in q$).

Lemma 2.9. *Let $\vec{x} \in V$, let $\vec{a} = (a_1,a_2,\ldots,a_k)$ be a finite sequence in $\mathbb{N}$ and let $p \in T(\vec{x})$. Then for each $\ell \in \mathbb{N}, MT(\vec{a},\langle S(\vec{x},n)\rangle_{n=\ell}^{\infty}) \in a_k \cdot p + a_{k-1} \cdot p + \ldots + a_1 \cdot p$.*

Proof. We prove the lemma (for all $\ell \in \mathbb{N}$) by induction on k. First assume $k = 1$ and let $\ell \in \mathbb{N}$ given. Since $p \in T(\vec{x})$ we have $FS(\langle S(\vec{x},n)\rangle_{n=\ell}^{\infty}) \in p$. Then $MT((a_1),\langle S(\vec{x},n)\rangle_{n=\ell}^{\infty}) = a_1 \cdot FS(\langle S(\vec{x},n)\rangle_{n=\ell}^{\infty}) \in a_1 \cdot p$.

Now let $k>1$ and assume the lemma has been established for all sequences of length $k-1$ and all $\ell\in\mathbb{N}$. Let $A=MT(\vec{a},\langle S(\vec{x},n)\rangle_{n=\ell}^{\infty})$. We show that $MT((a_1),\langle S(\vec{x},n)\rangle_{n=\ell}^{\infty})\subseteq\{x\in\mathbb{N}:A-x\in a_k\cdot p+\ldots+a_2\cdot p\}$, which will suffice. Let $z\in MT((a_1),\langle S(\vec{x},n)\rangle_{n=1}^{\infty})$ and pick a finite set $F_1\subseteq\{\ell,\ell+1,\ldots\}$ and for each $n\in F_1$, pick $w_n\in S(\vec{x},n)$ such that $z=a_1\cdot\sum_{n\in F_1}w_n$. Let $v=\max F_1+1$. Then by the induction hypothesis $MT((a_2,a_3,\ldots,a_k),\langle S(\vec{x},n)\rangle_{n=v}^{\infty})\in a_k\cdot p+\ldots+a_2\cdot p$, while $MT((a_2,a_3,\ldots,a_k),\langle S(\vec{x},n)\rangle_{n=v}^{\infty})\subseteq A-z$. ∎

Theorem 2.10. *Let $\vec{x}\in V^*$, let $\vec{a}=(a_1,a_2,\ldots,a_k)$ be a finite sequence in $\mathbb{N}$, let p be an idempotent in $T(\vec{x})$, and let $A\in a_k\cdot p+a_{k-1}\cdot p+\ldots+a_1\cdot p$. There exists $\vec{y}$ refining $\vec{x}$ such that $MT(\vec{a},\langle S(\vec{y},n)\rangle_{n=1}^{\infty})\subseteq A$. In particular, if $MT(\vec{a},\langle S(\vec{x},n)\rangle_{n=1}^{\infty})=\bigcup_{i=1}^{r}A_i$, then there exist some $\vec{y}$ refining $\vec{x}$ and some $i\in\{1,2,\ldots,r\}$ such that $MT(\vec{a},\langle S(\vec{y},n)\rangle_{n=1}^{\infty})\subseteq A_i$.*

Proof. We show first that the "In particular" statement follows. By [7, Lemma 2.4] $T(\vec{x})$ is a compact subsemigroup of $(\beta\mathbb{N},+)$ so by [4, Corollary 2.10] we may pick such an idempotent $p\in T(\vec{x})$. By Lemma 2.9, $\bigcup_{i=1}^{r}A_i\in a_k\cdot p+\ldots+a_1\cdot p$ so pick some $i\in\{1,2,\ldots,r\}$ such that $A_i\in a_k\cdot p+\ldots+a_1\cdot p$.

We now proceed to inductively construct $\vec{y}(n)$ for $n\in\mathbb{N}$, as well as a few auxiliary items needed for the induction. Specifically we choose $\vec{y}(n)\in\mathbb{N}^{m(n)}, D_n\in p, E_n\in p, g(n)\in\mathbb{N}, \tau(n)\in\mathbb{N}$, and for $t\in\mathbb{N}$ with $1\le t\le\min\{n,k\}, B_{n,t}\in a_k\cdot p+a_{k-1}\cdot p+\ldots+a_t\cdot p$ and $C_{n,t}\in a_t\cdot p$ satisfying the following induction hypotheses:

(1) $B_{n,1}=A$ and for $t\in\{2,3,\ldots,\min\{n,k\}\}$, $B_{n,t}=\bigcap\{B_{j,t-1}-a_{t-1}\cdot\sum_{\ell\in F}w_\ell : F\subseteq\{t-1,t,\ldots,n-1\}$ and $j=\min F$ and for all $\ell\in F, w_\ell\in S(\vec{y},\ell)\}$.

(2) For $t\in\{1,2,\ldots,\min\{n,k-1\}\}, C_{n,t}=\{x\in\mathbb{N}:B_{n,t}-x\in a_k\cdot p+\ldots+a_{t+1}\cdot p\}$ and if $n\ge k, C_{n,k}=B_{n,k}$.

(3) $D_n\subseteq C_{n,1}/a_1$ and if $n>1$,

$$D_n = D_{n-1}\cap\mathbb{N}2^{\tau(n-1)+1}\cap FS(\langle S(\vec{x},m)\rangle_{m=g(n-1)+1}^{\infty})\cap$$
$$\bigcap_{t=1}^{\min\{n,k\}}C_{n,t}/a_t\cap\bigcap\{D_{n-1}-w : w\in S(\vec{y},n-1)\}.$$

(4) $E_n=\{x\in\mathbb{N}:D_n-x\in p\}$

(5) $S(\vec{y},n)\subseteq D_n\cap E_n\cap FS(\langle S(\vec{x},m)\rangle_{m=g(n-1)+1}^{g(n)})$ and

(6) $\tau(n)=\max\{s\in\mathbb{N}:\exists a\in S(\vec{y},n)$ with $2^s\le a\}$.

To ground the induction let $g(0)=0$ let $B_{1,1}=A$ and let $C_{1,1}=\{x\in\mathbb{N}:B_{1,1}-x\in a_k\cdot p+\ldots+a_2\cdot p\}$ unless $k=1$ in which case $C_{1,1}=B_{1,1}$. Let $D_1=C_{1,1}/a_1$. Since $C_{1,1}\in a_1\cdot p, D_1\in p=p+p$. Let $E_1=\{x\in\mathbb{N}:D_1-x\in p\}$. Since $p\in U$ and $D_1\cap E_1\in p$, pick $\vec{y}(1)\in\mathbb{N}^{m(1)}$ such that $S(\vec{y},1)\subseteq D_1\cap E_1$. Pick $g(1)\in\mathbb{N}$ such that $S(\vec{y},1)\subseteq$

$FS(\langle S(\vec{x},m)\rangle_{m=1}^{g(1)})$ and let $\tau(1)=\max\{s\in\mathbb{N}$: there is some $a\in S(\vec{y},1)$ with $2^s\leq a\}$. All hypotheses are satisfied.

Now let $n>1$ and assume the construction has proceeded through $n-1$. Let $B_{n,t}$ and $C_{n,t}$ for $t\in\{1,2,\ldots,\min\{n,k\}\}$ be as required by hypotheses (1) and (2) and let D_n and E_n be as required by hypotheses (3) and (4).

Before proceeding to the choice of $\vec{y}(n)$, we show that these sets chosen above are where they are supposed to be. To do this we show first by induction on $|F|$ that if $F\subseteq\{1,2,\ldots,n-1\}, j=\min F$, and for each $\ell\in F, w_\ell\in S(\vec{y},\ell)$, then $\sum_{\ell\in F} w_\ell\in D_j$. If $F=\{j\}$ this follows from hypothesis (5) so assume $|F|>1$ and let $r=\min F\setminus\{j\}$. Then by our subsidiary induction hypothesis we have $\sum_{\ell\in F\setminus\{j\}} w_\ell\in D_r$. By hypothesis (3) $D_r\subseteq D_{j+1}\subseteq D_j-w_j$ so that $\sum_{\ell\in F} w_\ell\in D_j$ as claimed. Certainly $B_{n,1}\in a_k\cdot p+\ldots+a_1\cdot p$. Let $t\in\{2,3,\ldots,\min\{n,k\}\}$. Now $B_{n,t}$ is the intersection of finitely many sets of the form $B_{j,t-1}-a_{t-1}\cdot\sum_{\ell\in F} w_\ell$ so it suffices to show that each of these is in $a_k\cdot p+\ldots+a_t\cdot p$. To this end, let $F\subseteq\{t-1,t,\ldots,n-1\}$, let $j=\min F$, and for each $\ell\in F$, let $w_\ell\in S(\vec{y},\ell)$. As we have seen, then $\sum_{\ell\in F} w_\ell\in D_j$. By hypothesis (3) $D_j\subseteq C_{j,t-1}/a_{t-1}$ so $a_{t-1}\cdot\sum_{\ell\in F} w_\ell\in C_{j,t-1}$ so by hypothesis (2) $B_{n,t-1}-a_{t-1}\cdot\sum_{\ell\in F} w_\ell\in a_k\cdot p+\ldots+a_t\cdot p$ as required. Given $t\in\{1,2,\ldots,\min\{n,k\}\}$ we have $B_{n,t}\in a_k\cdot p+\ldots+a_t\cdot p$ so one concludes immediately that $C_{n,t}\in a_t\cdot p$.

Now $D_{n-1}\in p$. Since $p=p+p, \mathbb{N}2^{\tau(n-1)+1}\in p$. In fact, for any $\ell\in\mathbb{N}$, we have $\mathbb{N}\ell\in p$. The easiest way to see this is to consider the congruence classes mod ℓ. Since $p\in T(\vec{x})$, we have $FS(\langle S(\vec{x},m)\rangle_{m=g(n-1)+1}^{\infty})\in p$. Given $t\in\{1,2,\ldots,\min\{n,k\}\}, C_{n,t}\in a_t\cdot p$ so $C_{n,t}/a_t\in p$. Since $S(\vec{y},n-1)\subseteq E_{n-1}$ by hypothesis (5) we have $D_{n-1}-w\in p$ for each $w\in S(\vec{y},n-1)$. Consequently $D_n\in p$. Since $D_n\in p=p+p, E_n\in p$.

Now $D_n\cap E_n\in p$ and $p\in U$, so pick $\vec{y}(n)\in\mathbb{N}^{m(n)}$ with $S(\vec{y},n)\subseteq D_n\cap E_n$.

As $D_n\subseteq FS(\langle S(\vec{x},m)\rangle_{m=g(n-1)+1}^{g(n)})$ pick $g(n)$ with

$$S(\vec{y},n)\subseteq FS(\langle S(\vec{x},m)\rangle_{m=g(n-1)+1}^{g(n)}).$$

Let $\tau(n)$ be as required by condition (6).

The induction being complete observe that the choice of $\tau(n)$ guarantees that $\vec{y}\in V^*$ and the choice of $g(n)$ guarantees that $\vec{y}$ refines $\vec{x}$. To complete the proof we want to show that $MT(\vec{a},\langle S(\vec{y},n)\rangle_{n=1}^{\infty})\subseteq A$. To this end, let $F_1,F_2,\ldots,F_k$ be finite nonempty subsets of $\mathbb{N}$ with $\max F_t<\min F_{t+1}$, for $t\in\{1,2,\ldots,k-1\}$ and for each $\ell\in\cup_{t=1}^k F_t$ pick $w_\ell\in S(\vec{y},\ell)$. Let $n(t)=\min F_t$ and note that $n(t)\geq t$. We show by downward induction on $t\in\{1,2,\ldots,k\}$ that $a_k\cdot\sum_{\ell\in F_k} w_\ell+\ldots+a_t\cdot\sum_{\ell\in F_t} w_\ell\in$

$B_{n(t),t}$. For $t=k$ we have by hypothesis (3) that $D_{n(k)} \subseteq C_{n(k),k}/a_k$. As we have previously established $\sum_{\ell\in F_k} w_\ell \in D_{n(k)}$, so $a_k \cdot \sum_{\ell\in F_k} w_\ell \in C_{n(k),k} = B_{n(k),k}$. Now let $t\in\{2,3,\ldots,k\}$ and assume $a_k \cdot \sum_{\ell\in F_k} w_\ell + \ldots + a_t \cdot \sum_{\ell\in F_t} w_\ell \in B_{n(t)},t$. By hypothesis (1) $B_{n(t),t} \subseteq B_{n(t-1),t-1} - a_{t-1}\cdot \sum_{\ell\in F_{t-1}} w_\ell$, so $a_k \cdot \sum_{\ell\in F_k} w_\ell + \ldots + a_{t-1}\cdot \sum_{\ell\in F_{t-1}} w_\ell \in B_{n(t-1),t-1}$ as required. Thus we have $a_k \cdot \sum_{\ell\in F_k} w_\ell + \ldots + a_1 \cdot \sum_{\ell\in F_1} w_\ell \in B_{n(1),1} = A$. ∎

Recall that we are investigating the question of whether for any finite partition of $\mathbb{N}$ one cell contains solutions to all partition regular infinite matrices. Before providing a strong negative answer to this question in the next section we close this section with two results which establish a certain amount of uniformity among the Milliken-Taylor (m,p,c)-systems.

Corollary 2.11. *Let B be a finite set of finite sequences in $\mathbb{N}$ and let $\mathbb{N}=\cup_{i=1}^{r} A_i$. There exists $\vec{x}\in V^*$ such that for each $\vec{a}\in B$ there is some $i\in\{1,2,\ldots,r\}$ with $MT(\vec{a},\langle S(\vec{x},n)\rangle_{n=1}^{\infty})\subseteq A_i$.*

Proof. Pick any $\vec{y}_0\in V^*$ and apply Theorem 2.10 $|B|$ times, in each case refining the previously chosen sequence. ∎

Recall that a subset A of $\mathbb{N}$ is "large" if and only if for each m,p,c in $\mathbb{N}$ there is some $\vec{x}\in\mathbb{N}^m$ such that $S(m,p,c,\vec{x})\subseteq A$. Thus $U=\{q\in\beta\mathbb{N}$: for each $A\in q, A$ is large $\}$.

Theorem 2.12. *Let $\vec{a}=(a_1,\ldots,a_k)$ be a finite sequence in $\mathbb{N}$ and let $\mathbb{N}=\cup_{i=1}^{r}A_i$. Then there exists $i\in\{1,2,\ldots,r\}$ such that*
(1) A_i is large and
(2) there is some $\vec{x}\in V^*$ with $MT(\vec{a},\langle S(\vec{x},n)\rangle_{n=1}^{\infty})\subseteq A_i$.

Proof. By [2, Lemma 2] U is a compact subsemigroup of $(\beta\mathbb{N},+)$ and by [7, Lemma 2.4] so is $T(\vec{x})$. So pick by [4, Corollary 2.10] an idempotent p in $T(\vec{x})$. By [1, Lemma 2.2], U is an ideal of $(\beta\mathbb{N},\cdot)$ so that each $a_t\cdot p\in U$. Consequently, $a_k\cdot p + a_{k-1}\cdot p + \ldots + a_1\cdot p \in U$. Pick $i\in\{1,2,\ldots,r\}$ such that $A_i\in a_k\cdot p+\ldots+a_1\cdot p$. Since $a_k{\cdot}p+\ldots+a_1{\cdot}p\in U$, conclusion (1) holds while conclusion (2) holds by Theorem 2.10. ∎

It is a consequence of results in the next section that one cannot strengthen Theorem 2.12 to read "A_i contains an (m,p,c)-system", since $FS(\langle S(\vec{x},n)\rangle_{n=1}^{\infty}) = MT((1),\langle S(\vec{x},n)\rangle_{n=1}^{\infty})$.

3. Separating Milliken–Taylor Systems

We restrict our attention here to ordinary Milliken-Taylor systems. That is, sets of the form $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})$ (Definition 2.3).

Definition 3.1.
(a) $S=\{\vec{a}:\vec{a}$ is a finite sequence in $\mathbb{N}\}$.
(b) The function $c:S\to S$ deletes any consecutive repeated terms. (So $c((1,3,1,1,2,2,2))=(1,3,1,2)$.)
(c) An element $\vec{a}$ of S is "compressed" if and only if $\vec{a}=c(\vec{a})$.
(d) An equivalence relation $\approx$ on S is defined by $\vec{a}\approx\vec{b}$ if and only if there is a (positive) rational α such that $\alpha\cdot c(\vec{a})=c(\vec{b})$.

Theorem 3.2. *Let $\vec{a},\vec{b}\in S$ and assume $\vec{a}\approx\vec{b}$. Then whenever $\mathbb{N}=\bigcup_{i=1}^{r}A_i$, there exist $i\in\{1,2,\ldots,r\}$ and two sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ with $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})\subseteq A_i$ and $MT(\vec{b},\langle y_n\rangle_{n=1}^{\infty})\subseteq A_i$.*

Proof. Observe that $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})\subseteq MT(c(\vec{a}),\langle x_n\rangle_{n=1}^{\infty})$ for any sequence $\langle x_n\rangle_{n=1}^{\infty}$. (If $a_t=a_{t+1}$ then $a_{t+1}\cdot\sum_{\ell\in F_{t+1}}x_\ell+a_t\cdot\sum_{\ell\in F_t}x_\ell=a_t\cdot\sum_{\ell\in F_t\cup F_{t+1}}x_\ell$). Consequently we may presume $\vec{a}$ and $\vec{b}$ are compressed and hence that for some positive rational α we have $\alpha\vec{a}=\vec{b}$. Pick $m,r\in\mathbb{N}$ such that $\alpha=\frac{m}{r}$ and let $\vec{d}=m\vec{a}$. Pick by Theorem 2.5 some sequence $\langle z_n\rangle_{n=1}^{\infty}$ and some $i\in\{1,2,\ldots,r\}$ with $MT(\vec{d},\langle z_n\rangle_{n=1}^{\infty})\subseteq A_i$. For each n let $x_n=m\cdot z_n$ and let $y_n=r\cdot z_n$. Then $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})=MT(\vec{b},\langle y_n\rangle_{n=1}^{\infty})=MT(\vec{d},\langle z_n\rangle_{n=1}^{\infty})$. ∎

The rest of this section is devoted to a proof of the converse of Theorem 3.2, which we state below.

Theorem 3.3. *Let $\vec{a},\vec{b}\in S$ and assume that whenever $r\in\mathbb{N}$ and $\mathbb{N}=\bigcup_{i=1}^{r}A_i$, there exist $i\in\{1,2,\ldots,r\}$ and sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ with $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})\subseteq A_i$ and $MT(\vec{a},\langle y_n\rangle_{n=1}^{\infty})\subseteq A_i$. Then $\vec{a}\approx\vec{b}$.*

The proof will include a sequence of lemmas. We start by noting that we may restrict our attention to compressed sequences.

Lemma 3.4. *Let $\vec{a}\in S$ and let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in $\mathbb{N}$. There exists $\langle y_n\rangle_{n=1}^{\infty}$ such that $MT(c(\vec{a}),\langle y_n\rangle_{n=1}^{\infty})\subseteq MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})$.*

Proof. Let ℓ be the largest integer such that there is some t with $a_t=a_{t+1}=\ldots=a_{t+\ell-1}$. For each n let $y_n=\sum_{i=1}^{\ell}x_{\ell n+i}$. ∎

As a consequence of Lemma 3.4 and the fact that always $MT(\vec{a},\langle x_n\rangle_{n=1}^{\infty})\subseteq MT(c(\vec{a}),\langle x_n\rangle_{n=1}^{\infty})$ it suffices to restrict our attention to compressed sequences.

We will adopt the common "chromatic" terminology and talk about a finite "coloring" of $\mathbb{N}$ rather than dividing $\mathbb{N}$ into finitely many classes. Our construction is somewhat complex, so we start with an informal presentation of a special case, namely when the compressed sequences $\vec{a}$ and $\vec{b}$, called patterns in the following, consist of powers of 2.

Take a pattern, i.e. a compressed sequence, say 1, 2, 4.

The length (3, here) will be n. We will start by putting on more and more colors in a coloring, so that more and more is forced on a sequence $\langle x_i \rangle_{i=1}^{\infty}$ for which the Milliken-Taylor system generated by the pattern and $\langle x_i \rangle_{i=1}^{\infty}$ is monochromatic, and then at the end we will distinguish two arbitrary patterns of the form $2^{a_1}, \ldots, 2^{a_m}$.

When a number is written in binary, its *start* will be the position of its most significant 1 (the unit digit is position 0, etc.), its *end* will be the position of its least significant 1, and its *gap sequence* will be the sequence of gaps between consecutive 1s. For example, the number 11001000100 has start 10, end 2, and gap sequence 0,2,3.

Pick a number k, much larger than a, b, c and n. Suppose we have a sequence $x_1, x_2, \ldots$ such that all the numbers formed by the pattern 1,2,4 on the x_i have the same color in some given coloring. By taking linear combinations, we may as well assume that each x_i starts way to the left of x_{i-1}. Say the end of x_i is at least k greater than the start of x_{i-1}. From now on, any reference to an x_i will be restricted to $i > n$, so that each x_i could occur in a sum with coefficient 1,2 or 4.

Color by start mod k and end mod k (i.e., we have k^2 colors so far). Then all the $4x_i$ start in the same place mod k (as $i > n$), so all the x_i start in the same place mod k. Similarly, all the x_i end in the same place mod k. Thus the gap between an x_i and an x_{i-1} is fixed mod k: say it is g (so, mod k, g is the difference between the end of x_i and the start of x_{i-1}, with 1 subtracted).

Now color by the number, mod k, of gaps of length congruent to j mod k, for each $0 \leq j < k$. So we have k^k new colors (and hence k^{k+2} colors in total). Given a number y formed from the pattern (in other words, belonging to the Milliken-Taylor system generated by the pattern and $\langle x_i \rangle_{i=1}^{\infty}$), say ending with $4x_{i-1}$ (i.e. the largest x in the sum forming y is x_{i-1}, with of course coefficient 4), we are free to add $4x_i$ or not, as we please - both y and $y + 4x_i$ must have the same color. Now, adding $4x_i$ to y puts in some new gaps: one of length g (mod k), and also all the gaps inside x_i. We conclude that the distribution of gap-lengths (mod k) inside each x_i is the same, namely: the number of gaps in the gap sequence of x_i which are congruent to j mod k is -1 if $j \equiv g$ mod k, and 0 otherwise ('-1' and '0' are meant mod k).

We remark that at this stage we can already distinguish various patterns. For example, what is the gap-distribution for our pattern 1,2,4? Any y in that pattern must have gap-distribution as follows: -3 gaps of length g mod k, 2 gaps of length $g+1$ mod k, and no gap of length each other j mod k. So, having colored $\mathbb{N}$ like this, and found our x_i, we can read off the value of g - it is that value mod k for which the number of gaps congruent to it is -3. Thus to distinguish 1,2,4 from 2,4,16, say, we pick our large k (larger than the numbers in both patterns), color $\mathbb{N}$ as above, and find our alleged x_i for 1,2,4 and x_i' for 2,4,16. Then the distribution vector for the pattern on the x_i is of the form $(0, 0, \ldots, 0, -3, 2, 0, \ldots, 0)$ (where the jth entry in this vector is supposed to denote the number of gaps of length j mod k). However, the distribution vector for the other pattern is of the form $(0, 0, \ldots, 0, -3, 1, 1, 0, \ldots, 0)$. No vector is of both these forms, so we are done.

Of course, we cannot yet distinguish 1,2,8 from 1,4,8, and we certainly cannot distinguish 1,2,1,4,1 from 1,4,1,2,1. So we proceed to more colors. The next bit will do both (i.e., we won't have some more colors just to distinguish 1,2,8 from

1,4,8—we'll go for the whole 1,2,1,4,1 versus 1,4,1,2,1). One little remark: if we added colors to see where (mod k) each gap started (how many gaps of length j start in position h, and so on), we would indeed be able to distinguish 1,2,8 from 1,4,8. However, we would still not be able to distinguish 1,2,1,4,1 from 1,4,1,2,1, and it just happens that the colors we will use for this do not need to involve the start position.

Introduce new colors as follows, keeping all the old colors as well. For each ordered pair (h,j) of numbers mod k, we count how many times in the gap sequence of x (the number we are coloring) we have an h followed later by a k, which gives k^{k^2} new colors. Note that we do not insist that the j must be immediately after the h, and each h and j can be counted lots of times. For example, if x has gap sequence (2,1,3,2,7,1,7) then the ordered pair (2,1) occurs 3 times, while the ordered pair (3,7) occurs twice, and the pair (2,2) occurs once.

What happens when we consider a y in the pattern, say ending with $4x_{i-1}$? We know that y and $y+4x_i$ must have the same color. But what have we done to the ordered-pair-counts when we put on this $4x_i$? Thinking of $y+4x_i$ as being made up of $4x_i$, then a "dividing gap" (of length g) between the $4x_i$ and the y, and then the y, we have three places new gap-pairs could come in: (A) pairs of gaps both in the $4x_i$, (B) pairs of gaps of which the first is in the $4x_i$ and the second is the dividing gap, and (C) pairs of gaps of which the first is either in $4x_i$ or the dividing gap and the second is in the y. Now, (C) contributes nothing at all to any pair (h,j), for the simple reason that the gaps of x_i, together with one g, have distribution vector $(0,\ldots,0)$, as we saw a few paragraphs ago. (B) is easy: it adds -1 copies of (g,g) and 0 copies of everything else (again, this is by the distribution vector of x_i). We conclude that the pair-counts inside x_i must be 1 for (g,g) and 0 for each other (h,j).

So each x_i has gap count of -1 gaps of length g and 0 of each other length, and we also know that each x_i has gap-pair count of 1 ordered pair of gaps of lengths (g,g) and 0 of each other pair of lengths. Thus, for example, a pattern element formed from the pattern $(1,2)$ will have gap-pairs counts as follows (for some g): 3 of type (g,g), -1 of type $(g,g+1)$, -1 of type $(g+1,g)$ and 0 each other type. Note also that in a pattern with $n=3$, for example 1,8,2, the only gap-pair count which is not 0 and is not of a type (a,g) or (g,b) for any a and b is the type $(g-2,g+3)$, which gets a count of 1.

Now we look at triples (ordered triples of gaps). Just as above, when we add in colors for these counts we'll get that the triples-distribution of an x_i is -1 of type (g,g,g) and 0 of each other type. Also, in a pattern of $n=4$, like 1,2,8,1, the only triple which does not contain a g and does not get count 0 is the triple $(-3,2,1)$, which gets a count of 1.

Then on to quadruples, where each x_i will have 1 of type (g,g,g,g) and 0 of each other type, and so on. Thus, in total, to distinguish any two patterns (of powers of 2), say the patterns $2^{a_1},\ldots,2^{a_m}$ and $2^{b_1},\ldots,2^{b_n}$, where $m\leq n$, we do the following:

Pick a k much larger than all the 2^{a_i} and 2^{b_i} and n. Color $\mathbb{N}$ by start, end, gap distribution, gap-pair distribution, and so on up to $(n-1)$-tuple distribution all mod k. Suppose that we have x_i for the first pattern and x'_i for the second pattern so that both patterns yield the same color class. We can assume that each x_i starts far to the left to x_{i-1}, and similarly for the x'_i. Then the distribution of gaps of y, a typical element of the first pattern, has $-n$ in exactly one place (that of length g, say), and all other coefficients are between 0 and $n-1$. Similarly for y', a typical member of the second pattern. Thus $m=n$.

So now we also know g. There is a unique $(n-1)$-tuple (in the distribution of y) which does not contain a g and does not get coefficient 0, namely $(g+a_n-a_{n-1}, g+a_{n-1}-a_{n-2}, \ldots, g+a_2-a_1)$. Hence, as this must be the same for y', we have $a_i-a_{i-1}=b_i-b_{i-1}$ for every i. Thus one pattern is a multiple of the other, as required.

Now we will turn to the general case, and present a much more formal proof of Theorem 3.3.

We assume we have compressed sequences $\vec{a} = (a_1,a_2,\ldots,a_n)$ and $\vec{b} = (b_1,b_2,\ldots,b_m)$ with the property that whenever $\mathbb{N}$ is finitely colored, there exist sequences $\langle x_i\rangle_{i=1}^{\infty}$ and $\langle x'_i\rangle_{i=1}^{\infty}$ with $MT(\vec{a},\langle x_i\rangle_{i=1}^{\infty})\cup MT(\vec{b},\langle x'_i\rangle_{i=1}^{\infty})$ monochrome (i.e., all elements get the same color). We assume without loss of generality that $m\leq n$. (We shall in fact show in Lemma 3.7 that $m=n$).

Pick a prime $p>\max\{a_ib_j : i\in\{1,2,\ldots,n\} \text{ and } j\in\{1,2,\ldots,m\}\}$.

Let $k=2n+1$. We write $\mathbb{Z}_k=\{-n,-n+1,\ldots,0,1,\ldots,n-1,n\}$ (The idea is we want to be able to talk about negative members of $\mathbb{Z}_k$).

Now we introduce the notion of "gap", based very loosely on our power of 2 construction.

Given $\vec{c}=(d,e)\in\{1,2,\ldots,p-1\}\times\{p,p+1,\ldots,p^2-1\}$, pick $u\in\{1,2,\ldots,p-1\}$ and $v\in\{0,1,\ldots,p-1\}$ such that $e=u\cdot p+v$. Let $x\in\mathbb{N}$. Then a $\vec{\mathbf{c}}$*-gap* of x is an occurrence of $d0\ldots0uv$ in the base p-expansion of x. (It does not matter whether one requires at least one 0 between d and u, although of course one should make up one's mind. We shall require a 0.) The *location* of the gap is the location of d. More formally, given $x=\sum_{t=1}^{\ell}\alpha_t p^{i(t)}$ where $i(1)<i(2)<\ldots<i(\ell)$ and each $\alpha_t\in\{1,2,\ldots,p-1\}$, a $\vec{c}$-gap of x occurs at location $i(t)$ if only and if $\alpha_t=d$, $i(t-1)<i(t)-1$, $\alpha_{t-1}=u$ and one of (1) $t>2, i(t-2)=i(t-1)-1$, and $\alpha_{t-2}=v$, or (2) $t>2, i(t-2)<i(t-1)-1$, and $v=0$, or (3) $t=2$, $i(1)>0$ and $v=0$.

Now let $P=\{1,2,\ldots,p-1\}\times\{p,p+1,\ldots,p^2-1\}$. For each $t\in\{1,2,\ldots,n-1\}$ we define a function $\varphi_t:\mathbb{N}\times P^t\to\mathbb{Z}_k$ as follows. Given $\vec{c}\in P$ and $x\in\mathbb{N}, \varphi_1(x,\vec{c})$ is the number, mod k, of $\vec{c}$-gaps of x. More generally, given $t\in\{1,2,\ldots,n-1\}, x\in\mathbb{N}$, and $\vec{c}_1,\vec{c}_2,\ldots,\vec{c}_t\in P$, $\varphi_t(x,\vec{c}_1,\vec{c}_2,\ldots,\vec{c}_t)$ is the number, mod k, of ordered t-tuples $(\alpha_1,\alpha_2,\ldots,\alpha_t)\in\mathbb{N}^t$ such that $\alpha_1<\alpha_2<\ldots<\alpha_t$ and for each $i\in\{1,2,\ldots,t\}$, a $\vec{c}_i$-gap occurs at location α_i in the base p expansion of x.

Next we define functions $\lambda:\mathbb{N}\to\{p,p+1,\ldots,p^2-1\}$ and $\rho:\mathbb{N}\to\{1,2,\ldots,p-1\}$. Given $x\in\mathbb{N}, \lambda(x)$ represents the leftmost two significant digits of x, and $\rho(x)$ is the rightmost nonzero digit of x, all in the base p expansion. Thus, if $c\in\{1,2,\ldots,p-1\}$ and $d\in\{0,1,\ldots,p-1\}$ and for some $r\in\mathbb{N}_0, cp^r+dp^{r-1}\leq x<cp^r+(d+1)p^{r-1}$, then $\lambda(x)=cp+d$. If $c\in\{1,2,\ldots,p-1\}$ and $d,r\in\mathbb{N}_0$ and $x=p^r(d\cdot p+c)$, then $\rho(x)=c$.

Now we are prepared to define our coloring of $\mathbb{N}$. Given x and y in $\mathbb{N}$, agree that x and y get the same color if and only if $\rho(x)=\rho(y)$ and for each $t\in\{1,2,\ldots,n-1\}$

and each $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t \in P, \varphi_t(x, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = \varphi_t(y, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t)$. (Note that there are $(p-1)\cdot k^\alpha$ colors where $\alpha = \sum_{t=1}^{n-1} p^t(p-1)^{2t}$.)

We choose sequences $\langle x_i\rangle_{i=1}^\infty$ and $\langle x_i'\rangle_{i=1}^\infty$ such that $MT(\vec{a}, \langle x_i\rangle_{i=1}^\infty) \cup MT(\vec{b}, \langle x_i'\rangle_{i=1}^\infty)$ is monochrome. By Theorem 2.5 applied to $FS(\langle x_i\rangle_{i=1}^\infty) = MT((1), \langle x_i\rangle_{i=1}^\infty)$ and $FS(\langle x_i'\rangle_{i=1}^\infty)$ we may refine the sequence so that (using the same names for the refined sequences) for each $j \in \{1,2,\ldots,n\}, FS(\langle a_j x_i\rangle_{i=1}^\infty)$ is monochrome with respect to our given coloring and with respect to λ and for each $j \in \{1,2,\ldots,m\}, FS(\langle b_j x_i'\rangle_{i=1}^\infty)$ is monochrome with respect to our given coloring and with respect to λ. Taking one further refinement we can also assume that for each $i \in \mathbb{N}$ and $r \in \mathbb{N}_0$, if $p^r \le x_i$, then p^{r+3} divides x_{i+1} and similarly if $p^r \le x_i'$, then p^{r+3} divides x_i'. This guarantees that for any $j, s \in \{1,2,\ldots,n\}$ there will be at least one 0 between the rightmost nonzero digit of $a_j x_{i+1}$ and the leftmost nonzero digit of $a_s x_i$. We may do this since given x_i some p^{r+3} terms of $\langle x_j\rangle_{j=i+1}^\infty$ must be congruent to 0 mod p^{r+3} and hence their sum is divisible by p^{r+3}.

Define functions $f : \{1,2,\ldots,n\} \to \{p, p+1, \ldots, p^2-1\}, f' : \{1,2,\ldots,m\} \to \{p, p+1, \ldots, p^2-1\}, g : \{1,2,\ldots,n\} \to \{1,2,\ldots,p-1\}$, and $g' : \{1,2,\ldots,m\} \to \{1,2,\ldots,p-1\}$ by $f(j) = \lambda(a_j x_1)$, $f'(j) = \lambda(b_j x_1')$, $g(j) = \rho(a_j x_1)$, and $g'(j) = \rho(b_j x_1')$. For $j \in \{1,2,\ldots,n\}$, let $\vec{d}_j = (g(j), f(j))$ and for $j \in \{1,2,\ldots,m\}$ let $\vec{d}'_j = (g'(j), f'(j))$. For $j \in \{1,2,\ldots,n-1\}$ let $\vec{e}_j = (g(j+1), f(j))$ and for $j \in \{1,2,\ldots,m-1\}$, let $\vec{e}'_j = (g'(j+1), f'(j))$. Thus there is a $\vec{d}_j$-gap in $a_j x_2 + a_j x_1$ and an $\vec{e}_j$-gap in $a_{j+1}x_2 + a_j x_1$.

Several of the following lemmas will be stated in terms of $\vec{a}, \langle x_i\rangle_{i=1}^\infty, f, g, \vec{d}_j$ and $\vec{e}_j$, corresponding lemmas for $\vec{b}, \langle x_i'\rangle_{i=1}^\infty, f', g', \vec{d}'_j$, and $\vec{e}'_j$ are of course also valid and we will feel free to utilize them.

Lemma 3.5. *Let $s, j \in \{1,2,\ldots,n\}$ with $s < n$. Then $\vec{e}_s \neq \vec{d}_j$.*

Proof. Suppose $\vec{e}_s = \vec{d}_j$. Then $g(s+1) = g(j)$ and $f(s) = f(j)$. That is $\rho(a_{s+1}x_1) = \rho(a_j x_1)$ and $\lambda(a_s x_1) = \lambda(a_j x_1)$. Now $a_j, a_{s+1} \in \{1,2,\ldots,p-1\}$ and computation of the rightmost nonzero digit of a product is merely multiplication mod p, so by cancellation we have $a_{s+1} = a_j$.

We now claim that since $\lambda(a_s x_1) = \lambda(a_j x_1)$, we have $a_s = a_j$. To see this, let r be the position of the leftmost significant digit of x_1. That is, $p^r \le x_1 < p^{r+1}$. Let $cp + d = \lambda(a_s x_1) = \lambda(a_j x_1)$, where $c \in \{1,2,\ldots,p-1\}$ and $d \in \{0,1,\ldots,p-1\}$. Assume first that $a_s \le a_j$. If equality holds we have established our claim, so assume $a_s < a_j$. Note that $p^r \le a_s x_1 < a_j x_1 < p^{r+2}$.

Case 1. $a_j x_1 < p^{r+1}$. Then (ignoring the second digit) we have $cp^r \le a_s x_1 < a_j x_1 < (c+1)p^r$, so $a_j x_1 < (c+1)p^r = cp^r + p^r \le a_s x_1 + p^r$ so $x_1 \le (a_j - a_s)x_1 < p^r$, a contradiction.

Case 2. $a_s x_1 < p^{r+1} < a_j x_1$. Then $cp^r + dp^{r-1} \leq a_s x_1 < cp^r + (d+1)p^{r-1}$ and $cp^{r+1}+dp^r \leq a_j x_1 < cp^{r+1}+(d+1)p^r$ so $a_j x_1 \geq cp^{r+1}+dp^r = p\cdot(c\cdot p^r+(d+1)\cdot p^{r-1})-p^r > p\cdot a_s x_1 - p^r$. Then $p^r > (p\cdot a_s - a_j)x_1 \geq x_1 \geq p^r$, a contradiction. (Note $p\cdot a_s \geq p > a_j$, so $p\cdot a_s - a_j \geq 1$.)

Case 3. $p^{r+1} < a_s x_1$. Then $cp^{r+1}+dp^r \leq a_s x_1 < a_j x_1 < cp^{r+1}+(d+1)p^r$ so $a_j x_1 < cp^{r+1}+dp^r+p^r \leq a_s x_1 + p^r \leq a_s x_1 + x_1 \leq a_j x_1$, a contradiction.

The case $a_s \geq a_j$ is handled the same way.

Thus we have established that $a_s = a_j$. Since we saw earlier that $a_{s+1} = a_j$, we have $a_s = a_{s+1}$, contradicting the fact that $\vec{a}$ is a compressed sequence. ∎

We first consider just gaps—later we will go on to pairs of gaps, triples of gaps and so on.

Lemma 3.6. *Let $\vec{c} \in P$ and let $j \in \{1,2,\ldots,n\}$. Then*

$$\varphi_1(a_j x_1, \vec{c}) = \begin{cases} -1 & \text{if } \vec{c} = \vec{d}_j \\ 0 & \text{otherwise}. \end{cases}$$

Proof. Let ℓ be the location of the rightmost nonzero digit of $a_j x_2$. There is a $\vec{d}_j$-gap located at ℓ. Consequently, we have

$$\varphi_1(a_j x_2 + a_j x_1, \vec{c}) = \begin{cases} \varphi_1(a_j x_2, \vec{c}) + \varphi_1(a_j x_1, \vec{c}) + 1 & \text{if } \vec{c} = \vec{d}_j \\ \varphi_1(a_j x_2, \vec{c}) + \varphi_1(a_j x_1, \vec{c}) & \text{if } \vec{c} \neq \vec{d}_j. \end{cases}$$

Since $\varphi_1(a_j x_2 + a_j x_1, \vec{c}) = \varphi_1(a_j x_2, \vec{c}) = \varphi_1(a_j x_1, \vec{c})$, the conclusion follows. ∎

Already from Lemma 3.6, we can read off information about our sequences. Let $\Delta = \{\vec{d}_j : j \in \{1,2,\ldots,n\}\}$ and let $\Delta' = \{\vec{d}'_j : j \in \{1,2,\ldots,m\}\}$.

Lemma 3.7. $\Delta = \Delta'$ *and* $m = n$.

Proof. Define $\Gamma : P \to \mathbb{Z}_k$ by $\Gamma(\vec{c}) = \varphi_1(a_n x_n + a_{n-1}x_{n-1} + \ldots + a_1 x_1, \vec{c})$ and note that also $\Gamma(\vec{c}) = \varphi_1(b_m x'_m + b_{m-1}x'_{m-1} + \ldots + b_1 x'_1, \vec{c})$, because $MT(\vec{a}, \langle x_i\rangle_{i=1}^{\infty}) \cup MT(\vec{b}, \langle x'_i\rangle_{i=1}^{\infty})$ is monochrome with respect to our given coloring. We show that $\Delta = \{\vec{c} \in P : \Gamma(\vec{c}) < 0\}$ and that $\sum_{\vec{c}\in\Delta} \Gamma(\vec{c}) = -n$. Since we can conclude similarly that $\Delta' = \{\vec{c} \in P : \Gamma(\vec{c}) < 0\}$ and $\sum_{\vec{c}\in\Delta'} \Gamma(\vec{c}) = -m$, we will be able to conclude that $n = m$.

For each $j \in \{1,2,\ldots,n-1\}$, let $\ell(j)$ be the location of the rightmost nonzero digit of $a_{j+1}x_{j+1}$. Then there is an $\vec{e}_j$-gap of $a_n x_n + a_{n-1}x_{n-1} + \ldots + a_1 x_1$ located at $\ell(j)$. Consequently, given any $\vec{c} \in P$,

$$\Gamma(\vec{c}) = \varphi_1(a_n x_n + a_{n-1}x_{n-1} + \ldots + a_1 x_1, \vec{c})$$

$$= \sum_{j=1}^{n} \varphi_1(a_j x_j, \vec{c}) + |\{j \in \{1,2,\ldots,n-1\} : \vec{c} = \vec{e}_j\}|$$

$$= \sum_{j=1}^{n} \varphi_1(a_j x_1, \vec{c}) + |\{j \in \{1,2,\ldots,n-1\} : \vec{c} = \vec{e}_j\}|.$$

If $\vec{c} \in \Delta$, then by Lemma 3.5, $\vec{c} \neq \vec{e}_j$ for any $j \in \{1,2,\ldots,n-1\}$ so $\Gamma(\vec{c}) = -|\{j \in \{1,2,\ldots,n\} : \vec{c} = \vec{d}_j\}|$ by Lemma 3.6.

If $\vec{c} = \vec{e}_j$ for some j, then $\vec{c} \notin \Delta$ by Lemma 3.5 so by Lemma 3.6, $\Gamma(\vec{c}) = |\{j \in \{1,2,\ldots,n-1\} : \vec{c} = \vec{e}_j\}|$. If $\vec{c} \notin \Delta \cup \{\vec{e}_j : j \in \{1,2,\ldots,n-1\}\}$, then by Lemma 3.6, $\Gamma(\vec{c}) = 0$. All conclusions follow now. ∎

We now turn our attention to t-tuples of gaps. In the next lemma we compute $\varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t)$ and $\varphi_t(a_j x_j + f(j), \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t)$. Note that the effect of adding $f(j)$ is to install a $\vec{d}_j$-gap at ℓ, where ℓ is the location of the rightmost digit of $a_j x_j$. This lemma is the key to the success of our construction.

Lemma 3.8. *Let $j \in \{1,2,\ldots,n\}$, let $t \in \{1,2,\ldots,n-1\}$ and let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t \in P$. Then $\varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t)$*

$$= \begin{cases} 0 \text{ if } (\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) \neq (\vec{d}_j, \vec{d}_j, \ldots, \vec{d}_j) \\ -1 \text{ if } (\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = (\vec{d}_j, \vec{d}_j, \ldots, \vec{d}_j) \text{ and } t \text{ is odd} \\ 1 \text{ if } (\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = (\vec{d}_j, \vec{d}_j, \ldots, \vec{d}_j) \text{ and } t \text{ is even} \end{cases}$$

and $\varphi_t(a_j x_j + f(j), \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = 0$.

Proof. We establish both statements by induction on t. For $t=1$, the first statement holds by Lemma 3.6. For the second we have that

$$\varphi_1(a_j x_j + f(j), \vec{c}) = \begin{cases} \varphi_1(a_j x_j) & \text{if } \vec{c} \neq \vec{d}_j \\ \varphi_1(a_j x_j) + 1 & \text{if } \vec{c} = \vec{d}_j. \end{cases}$$

Now let $t \in \{2,3,\ldots,n-1\}$ and assume both statements are valid for $t-1$. Let $E = \{(\alpha_1, \alpha_2, \ldots, \alpha_t) \in \mathbb{N}^t : \alpha_1 < \alpha_2 < \ldots < \alpha_t$ and for each $i \in \{1,2,\ldots,t\}$, a $\vec{c}_i$-gap of $a_j x_{j+1} + a_j x_j$ occurs at $\alpha_i\}$. Let ℓ be the location of the leftmost nonzero digit of $a_j x_j$ and let r be the lo cation of the rightmost nonzero digit of $a_j x_{j+1}$. Note that $r > \ell + 1$.

Let $A = \{(\alpha_1, \alpha_2, \ldots, \alpha_t) \in E : \alpha_t \leq \ell\}$.

For $i \in \{1,2,\ldots,t-1\}$, let $B_i = \{(\alpha_1, \alpha_2, \ldots, \alpha_t) \in E : \alpha_i \leq \ell$ and $\alpha_{i+1} \geq r\}$. Let $C = \{(\alpha_1, \alpha_2, \ldots, \alpha_t) \in E : \alpha_1 = r\}$ and let $D = \{(\alpha_1, \alpha_2, \ldots, \alpha_t) \in E : \alpha_1 > r\}$. Then the listed sets are pairwise disjoint and $E = A \cup \bigcup_{i=1}^{t-1} B_i \cup C \cup D$.

Now $|A| \equiv \varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t)$, where this and other congruences in this proof are always mod k. For $i \in \{1,2,\ldots,t-1\}, |B_i| \equiv \varphi_{t-i}(a_j x_{j+1} +$

$f(j), \vec{c}_{i+1}, \vec{c}_{i+2}, \ldots, \vec{c}_t) \cdot \varphi_i(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_i)$, for there is a $\vec{d}_j$-gap of $a_j x_{j+1} + a_j x_j$ located at r and there is a $\vec{d}_j$-gap of $a_j x_{j+1} + f(j)$ located at r. If $\vec{c}_1 \neq \vec{d}_j$, then $C = \emptyset$. If $\vec{c}_1 = \vec{d}_j$, then $|C| \equiv \varphi_{t-1}(a_j x_{j+1}, \vec{c}_2, \vec{c}_3, \ldots, \vec{c}_t)$. Moreover, $|D| = \varphi_t(a_j x_{j+1}, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t)$.

By the induction hypothesis we have for each $i \in \{1,2,\ldots,t-1\}$ that $|B_i| \equiv 0$, since

$$\varphi_{t-i}(a_j x_{j+1} + f(j), \vec{c}_{i+1}, \ldots, \vec{c}_t) = 0.$$

Then we have that if $\vec{c}_1 \neq \vec{d}_j$,

$$\begin{aligned} &\varphi_t(a_j x_{j+1} + a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) \\ &\qquad \equiv \varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) + \varphi_t(a_j x_{j+1}, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) \end{aligned}$$

and if $\vec{c}_1 = \vec{d}_j$, then

$$\begin{aligned} \varphi_t(a_j x_{j+1} + a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) &\equiv \varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) \\ &+ \varphi_{t-1}(a_j x_{j+1}, \vec{c}_2, \vec{c}_3, \ldots, \vec{c}_t) + \varphi_t(a_j x_{j+1}, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t). \end{aligned}$$

Since

$$\begin{aligned} \varphi_t(a_j x_{j+1} + a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) &= \varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) \\ &= \varphi_t(a_j x_{j+1}, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) \end{aligned}$$

and

$$\varphi_{t-1}(a_j x_{j+1}, \vec{c}_2, \vec{c}_3, \ldots, \vec{c}_t) = \varphi_{t-1}(a_j x_j, \vec{c}_2, \vec{c}_3, \ldots, \vec{c}_t),$$

we conclude that when $\vec{c}_1 \neq \vec{d}_j$, then $\varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = 0$ while if $\vec{c}_1 = \vec{d}_j$, $\varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = -\varphi_{t-1}(a_j x_j, \vec{c}_2, \ldots, \vec{c}_t)$. Thus the first statement holds.

To establish the second equation note that if $\vec{c}_i \neq \vec{d}_j$, then

$$\begin{aligned} &\varphi_t(a_j x_j + f(j), \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) \\ &= \varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = 0. \end{aligned}$$

Thus assume $\vec{c} = \vec{d}_j$. Then

$$\begin{aligned} &\varphi_t(a_j x_j + f(j), \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) \\ &= \varphi_{t-1}(a_j x_j, \vec{c}_2, \vec{c}_3, \ldots, \vec{c}_t) + \varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t). \end{aligned}$$

If $(\vec{c}_2, \vec{c}_3, \ldots, \vec{c}_t) \neq (\vec{d}_j, \vec{d}_j, \ldots, \vec{d}_j)$, then both of these last terms are 0. If $(\vec{c}_2, \vec{c}_3, \ldots, \vec{c}_t) = (\vec{d}_j, \vec{d}_j, \ldots, \vec{d}_j)$, one of them is 1 and the other is -1. ∎

From Lemma 3.8 we can read off the effect on $a_j x_j$ of adding $f(j-1)$.

Lemma 3.9. *Let $j \in \{2,3,\ldots,n\}$, let $t \in \{1,2,\ldots,n-1\}$ and let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t \in P$. Then $\varphi_t(a_j x_j + f(j-1), \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t)$*

$$= \begin{cases} 1 \text{ if } (\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = (\vec{d}_j, \vec{d}_j, \ldots, \vec{d}_j) \text{ and } t \text{ is even} \\ -1 \text{ if } (\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = (\vec{d}_j, \vec{d}_j, \ldots, \vec{d}_j) \text{ and } t \text{ is odd} \\ -1 \text{ if } (\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = (\vec{e}_{j-1}, \vec{d}_j, \ldots, \vec{d}_j) \text{ and } t \text{ is even} \\ 1 \text{ if } (\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = (\vec{e}_{j-1}, \vec{d}_j, \ldots, \vec{d}_j) \text{ and } t \text{ is odd} \\ 0 \text{ otherwise.} \end{cases}$$

Proof. Note that adding $f(j-1)$ installs an $\vec{e}_{j-1}$-gap at r where r is the location of the rightmost nonzero digit of $a_j x_j$. Then, if $\vec{c}_1 \neq \vec{e}_{j-1}$, we have that $\varphi_t(a_j x_j + f(t-1), \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = \varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t)$, while if $\vec{c}_1 = \vec{e}_{j-1}$ we have that $\varphi_t(a_j x_j + f(t-1), \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = \varphi_{t-1}(a_j x_j, \vec{c}_2, \vec{c}_3, \ldots, \vec{c}_t)$, since $\varphi_t(a_j x_j, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) = 0$. Then the conclusion follows from Lemma 3.8. ∎

We will apply the information we have learned in Lemmas 3.6 to 3.9, by starting with the case $t=1$.

Lemma 3.10. *Let $j \in \{1,2,\ldots,n\}$ and let $\vec{c} \in P$. Then $\varphi_1(a_j x_j + a_{j-1} x_{j-1} + \ldots + a_1 x_1, \vec{c})$*

$$= \begin{cases} -|\{i \in \{1,2,\ldots,j\} : \vec{c} = \vec{d}_i\}| \text{ if } \vec{c} \in \Delta \\ |\{i \in \{1,2,\ldots,j-1\} : \vec{c} = \vec{e}_i\}| \text{ if } \vec{c} = \vec{e}_i \text{ for} \\ \text{some } i \in \{1,2,\ldots,j-1\} \\ 0 \text{ otherwise.} \end{cases}$$

Proof. If $r(i)$ is the location of the rightmost nonzero digit of $a_{i+1} x_{i+1}$ then for each $i \in \{1,2,\ldots,j-1\}$, there is an $\vec{e}_i$-gap of $a_j x_j + a_{j-1} x_{j-1} + \ldots + a_1 x_1$ located at $r(i)$. Consequently, $\varphi_1(a_j x_j + a_{j-1} x_{j-1} + \ldots + a_1 x_1, \vec{c}) = \sum_{i=1}^{j} \varphi_1(a_i x_i, \vec{c}) + |\{i \in \{1,2,\ldots,j-1\} : \vec{c} = \vec{e}_i\}|$. Then Lemma 3.6 applies. ∎

We now turn to general t-tuples of gaps. In the following two lemmas we will see the importance of Lemma 3.8.

Lemma 3.11. *Let $j \in \{1,2,\ldots,n\}$, let $t \in \{1,2,\ldots,n-1\}$ and let $\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t \in P$.*
(a) If $\varphi_t(a_j x_j + a_{j-1} x_{j-1} + \ldots + a_1 x_1, \vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t) \neq 0$, then either
(1) $\{\vec{c}_1, \vec{c}_2, \ldots, \vec{c}_t\} \cap \Delta \neq \emptyset$, or
(2) There exist $1 \leq \ell(1) < \ell(2) < \ldots < \ell(t) < j$ such that for each $i \in \{1,2,\ldots,t\}$, $\vec{c}_i = \vec{e}_{\ell(i)}$.

(b) If (a)(2) holds and $t=j-1$, *then* $\varphi_t(a_jx_j+a_{j-1}x_{j-1}+\ldots+a_1x_1, \vec{c}_1, \vec{c}_2,\ldots, \vec{c}_t)=1$.

Proof. We proceed by induction on $\min\{j,t\}$. If $j=1$ (in which case $t\neq j-1$) this is a consequence of Lemma 3.8. If $t=1$ this is a consequence of Lemma 3.10. If (a)(2) holds and $t=j-1$, one has $j=2$ so $|\{i\in\{1,2,\ldots,j-1\}:\vec{c}=\vec{e}_i\}|=1$.

Now assume $\min\{j,t\}>1$ and the lemma is true for smaller values. We proceed as in the proof of Lemma 3.8. Let $E=\{(\alpha_1,\alpha_2,\ldots,\alpha_t)\in\mathbb{N}^t:\alpha_1<\alpha_2<\ldots<\alpha_t$ and for each $i\in\{1,2,\ldots,t\}$, a $\vec{c}_i$-gap of $a_jx_j+a_{j-1}x_{j-1}+\ldots+a_1x_1$ occurs at $\alpha_i\}$. Let ℓ be the location of the leftmost nonzero digit of $a_{j-1}x_{j-1}$ and let r be the location of the rightmost nonzero digit of a_jx_j. Let $A=\{(\alpha_1,\alpha_2,\ldots,\alpha_t)\in E:\alpha_t\leq\ell\}$.

For $i\in\{1,2,\ldots,t-1\}$, let $B_i=\{(\alpha_1,\alpha_2,\ldots,\alpha_t)\in E:\alpha_i\leq\ell$ and $\alpha_{i+1}\geq r\}$. Let $C=\{(\alpha_1,\alpha_2,\ldots,\alpha_t)\in E:\alpha_1=r\}$ and let $D=\{(\alpha_1,\alpha_2,\ldots,\alpha_t)\in E:\alpha_1>r\}$. Then the listed sets are pairwise disjoint and $E=A\cup\bigcup_{i=1}^{t-1}B_i\cup C\cup D$. Then $|A|\equiv\varphi_t(a_{j-1}x_{j-1}+a_{j-2}x_{j-2}+\ldots+a_1x_1, \vec{c}_1, \vec{c}_2,\ldots, \vec{c}_t)$. For $i\in\{1,2,\ldots,t-1\}, |B_i|\equiv\varphi_{t-i}(a_jx_j+f(j-1), \vec{c}_{i+1}, \vec{c}_{i+2},\ldots, \vec{c}_t)\cdot\varphi_i(a_{j-1}x_{j-1}+a_{j-2}x_{j-2}+\ldots+a_1x_1, \vec{c}_1, \vec{c}_2,\ldots, \vec{c}_i)$, since $a_jx_j+f(j-1)$ and $a_jx_j+a_{j-1}x_{j-1}+\ldots+a_1x_1$ both have an $\vec{e}_{j-1}$-gap at r.

If $\vec{c}_1\neq\vec{e}_{j-1}$, then we have that $|C|\equiv\varphi_t(a_jx_j, \vec{c}_1, \vec{c}_2,\ldots, \vec{c}_t)$, and if $\vec{c}_1=\vec{e}_{j-1}$, then $|C|\equiv\varphi_{t-1}(a_jx_j, \vec{c}_2, \vec{c}_3,\ldots, \vec{c}_t)+\varphi_t(a_jx_j, \vec{c}_1, \vec{c}_2,\ldots, \vec{c}_t)$. And $|D|\equiv\varphi_t(a_jx_j, \vec{c}_1, \vec{c}_2,\ldots, \vec{c}_t)$.

Assume first that $\varphi_t(a_jx_j+a_{j-1}x_{j-1}+\ldots+a_1x_1, \vec{c}_1, \vec{c}_2,\ldots, \vec{c}_t)\neq 0$ and that $\{\vec{c}_1, \vec{c}_2,\ldots, \vec{c}_t\}\cap\Delta=\emptyset$. Then by Lemma 3.8, $|C|\equiv 0$ and $|D|\equiv 0$. If $|A|\not\equiv 0$, we have by the induction hypothesis that there exist $1\leq\ell(1)<\ell(2)<\ldots<\ell(t)<j-1$ such that for each $i\in\{1,2,\ldots,t\}$, we have $\vec{c}_i=\vec{e}_{\ell(i)}$. Since then $\ell(t)<j$, conclusion (2) holds. Now assume $|A|\equiv 0$. Then for some $i\in\{1,2,\ldots,t-1\}$ we must have $|B_i|\not\equiv 0$ so that, in particular, $\varphi_{t-i}(a_jx_j+f(j-1), \vec{c}_{i+1}, \vec{c}_{i+2},\ldots, \vec{c}_t)\neq 0$. By Lemma 3.9 then $i=t-1$ and $\vec{c}_t=\vec{e}_{j-1}$. Then also $\varphi_{t-1}(a_{j-1}x_{j-1}+a_{j-2}x_{j-2}+\ldots+a_1x_1, \vec{c}_1, \vec{c}_2,\ldots, \vec{c}_{t-1})\neq 0$ so by the induction hypothesis there exist $1\leq\ell(1)<\ldots<\ell(t-1)<j-1$ such that for each $i\in\{1,2,\ldots,t-1\}, \vec{c}_r=\vec{e}_{\ell(i)}$. Letting $\ell(t)=j-1$, we see that conclusion (2) holds.

Turning to (b), assume that (2) holds and that $t=j-1$. Then since $1\leq\ell(1)<\ell(2)<\ldots<\ell(t)<j$ we have for each $i\in\{1,2,\ldots,t\}$ that $\ell(i)=i$. We have by Lemma 3.5 that $\{\vec{c}_1, \vec{c}_2,\ldots, \vec{c}_t\}\cap\Delta=\emptyset$. Then by Lemma 3.8 $|C|\equiv|D|\equiv 0$. If we had $|A|\not\equiv 0$ we would have by the induction hypothesis some $1\leq\ell'(1)<\ell'(2)<\ldots<\ell'(t)<j-1$ which is impossible since $t=j-1$. Further, if $i\in\{1,2,\ldots,t-2\}$ we have by Lemma 3.9 that $\varphi_{t-i}(a_jx_j+f(j-1), \vec{c}_{i+1}, \vec{c}_{i+2},\ldots, \vec{c}_t)=0$. Thus we must have $\varphi_t(a_jx_j+a_{j-1}x_{j-1}+\ldots+a_1x_1, \vec{c}_1, \vec{c}_2,\ldots, \vec{c}_t)=\varphi_1(a_jx_j+f(j-1), \vec{e}_{j-1})\cdot\varphi_{j-2}(a_{j-1}x_{j-1}+a_{j-2}x_{j-2}+\ldots+a_1x_1, \vec{e}_1, \vec{e}_2,\ldots, \vec{e}_{j-2})$. The first of these terms equals 1 by Lemma 3.9 and the second equals 1 by the induction hypothesis. ∎

Lemma 3.12. $(\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_n)=(\vec{e}\,'_1,\vec{e}\,'_2,\ldots,\vec{e}\,'_n)$ and $g(1)=g'(1)$.

Proof. First $g(1)=\rho(a_1x_1)=\rho(a_nx_n+a_{n-1}x_{n-1}+\ldots+a_1x_1)=\rho(b_nx'_n+b_{n-1}x'_{n-1}+\ldots+b_1x'_1)=\rho(b_1x'_1)=g'(1)$. By the dual of Lemma 3.11 we have $\varphi_{n-1}(b_nx'_n+b_{n-1}x'_{n-1}+\ldots+b_1x'_1,\vec{e}\,'_1,\vec{e}\,'_2,\ldots,\vec{e}\,'_{n-1})=1$.

Since $\varphi_{n-1}(a_nx_n+a_{n-1}x_{n-1}+\ldots+a_1x_1,\vec{e}\,'_1,\vec{e}\,'_2,\ldots,\vec{e}\,'_{n-1})=\varphi_{n-1}(b_nx'_n+b_{n-1}x'_{n-1}+\ldots+b_1x'_1,\vec{e}\,'_1,\vec{e}\,'_2,\ldots,\vec{e}\,'_{n-1})$, we have by Lemma 3.11 that either $\{\vec{e}\,'_1,\vec{e}\,'_2,\ldots,\vec{e}\,'_{n-1}\}\cap\Delta\neq\emptyset$ or $(\vec{e}\,'_1,\vec{e}\,'_2,\ldots,\vec{e}\,'_{n-1})=(\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_{n-1})$.

By the dual of Lemma 3.5, $\{\vec{e}\,'_1,\vec{e}\,'_2,\ldots,\vec{e}\,'_{n-1}\}\cap\Delta'=\emptyset$, while by Lemma 3.7, $\Delta=\Delta'$. Thus $(\vec{e}\,'_1,\vec{e}\,'_2,\ldots,\vec{e}\,'_{n-1})=(\vec{e}_1,\vec{e}_2,\ldots,\vec{e}_{n-1})$ as required. ∎

The following lemma completes the proof of Theorem 3.3.

Lemma 3.13. $\vec{a}\approx\vec{b}$.

Proof. By Lemma 3.12 we have that for each $j\in\{1,2,\ldots,n\},g(j)=g'(j)$. It suffices to show that for $j\in\{1,2,\ldots,n-1\}$ we have $b_{j+1}/a_{j+1}=b_j/a_j$ (for then, letting $\alpha=b_j/a_j$ we have $\vec{b}=\alpha\vec{a}$). So let $j\in\{1,2,\ldots,n-1\}$. Let $d=\rho(x_1)$ and let $d'=\rho(x'_1)$. Then (with congruence mod p) we have $b_jd'\equiv\rho(b_jx'_1)=g'(j)=g(j)=\rho(a_jx_1)\equiv a_jd$ and similarly $b_{j+1}d'\equiv a_{j+1}d$. Thus $a_{j+1}b_jd'\equiv a_{j+1}a_jd\equiv a_jb_{j+1}d'$ so by cancellation we have $a_{j+1}b_j\equiv a_jb_{j+1}$. Since $p>\max\{a_{j+1}b_j,a_jb_{j+1}\}$ we must then have that $a_{j+1}b_j=a_jb_{j+1}$ so that $b_j/a_j=b_{j+1}/a_{j+1}$. ∎

In the proof of Theorem 3.3 we used $(p-1)\cdot k^\alpha$ colors, where $\alpha=\sum_{t=1}^{n-1}p^t(p-1)^{2t}$.

In other words, to distinguish between $\vec{a}$ and $\vec{b}$, where the longer of the lengths of $\vec{a}$ and $\vec{b}$ is n and their maximum entry is M, we needed of the order of $n^{M^{6n}}$ colors. We now show that to distinguish $\vec{a}$ and $\vec{b}$, where $\vec{a}\not\approx\vec{b}$, two colors will do. In fact, more generally we have the following result:

Theorem 3.14. *Let $\vec{a}(1),\vec{a}(2),\ldots,\vec{a}(r)$ be finite sequences such that $\vec{a}(i)\not\approx\vec{a}(j)$ whenever $i\neq j$. Then there is a partition $\mathbb{N}=\cup_{i=1}^{r}A_i$ such that whenever $i,j\in\{1,2,\ldots,r\}$ and $\langle x_n\rangle_{n=1}^\infty$ is a sequence in $\mathbb{N}$ with $MT(\vec{a}(i),\langle x_n\rangle_{n=1}^\infty)\subseteq A_j$ one has $i=j$.*

Proof. By Theorem 3.3 there is for each $i\neq j$ a finite coloring which distinguishes between $MT(\vec{a}(i),\langle x_n\rangle_{n=1}^\infty)$ and $MT(\vec{a}(j),\langle y_n\rangle_{n=1}^\infty)$ whenever these are both monochrome. Consequently, by taking a common refinement, there is a partition $\mathbb{N}=\cup_{t=1}^{s}B_t$ such that there do not exist $i\neq j$ in $\{1,2,\ldots,r\},t\in\{1,2,\ldots,s\}$ and sequences $\langle x_n\rangle_{n=1}^\infty$ and $\langle y_n\rangle_{n=1}^\infty$ in $\mathbb{N}$ with $MT(\vec{a}(i),\langle x_n\rangle_{n=1}^\infty)\cup MT(\vec{a}(j),\langle y_n\rangle_{n=1}^\infty)\subseteq B_t$.

For each $i\in\{1,2,\ldots,r-1\}$ let $A_i=\cup\{B_t:t\in\{1,2,\ldots,s\}$ and there exists $\langle x_n\rangle_{n=1}^\infty$ with $MT(\vec{a}(i),\langle x_n\rangle_{n=1}^\infty)\subseteq B_t\}$. Let $A_r=\mathbb{N}\setminus\cup_{i=1}^{r-1}A_i$ and note that $A_r\supset\cup\{B_t:t\in\{1,2,\ldots,s\}$ and there exists $\langle x_n\rangle_{n=1}^\infty$ with $MT(\vec{a}(i),\langle x_n\rangle_{n=1}^\infty)\subseteq B_t\}$

(some B_t's may contain no Milliken-Taylor systems). Then $\mathbb{N}=\bigcup_{i=1}^{r} A_i$, so let $i,j\in\{1,2,\ldots,r\}$ and assume we have a sequence $\langle y_n\rangle_{n=1}^{\infty}$ with $MT(\vec{a}(i),\langle y_n\rangle_{n=1}^{\infty})\subseteq A_j$. Then $MT(\vec{a}(i),\langle y_n\rangle_{n=1}^{\infty})\subseteq\bigcup\{B_t : t\in\{1,2,\ldots,s\}$ and $B_t\subseteq A_j\}$. Thus by Corollary 2.6 we may pick $\langle x_n\rangle_{n=1}^{\infty}$ and $t\in\{1,2,\ldots,s\}$ such that $B_t\subseteq A_j$ and $MT(\vec{a}(i),\langle x_n\rangle_{n=1}^{\infty})\subseteq B_t$. Then $B_t\subseteq A_i$, so $i=j$. ∎

4. Separating Kernel Partition Regular Systems

In this final section we turn our attention to kernel partition regularity. Recall from the Introduction that a matrix A (with only finitely many non-zero entries on each row) is said to be kernel partition regular if, whenever $\mathbb{N}$ is partitioned into finitely many classes, there is a vector $\vec{x}$ with all its entries in the same class such that $A\vec{x}=\vec{0}$.

Our aim in this section is to exhibit kernel partition matrices A and B such that the diagonal matrix formed by A and B is not kernel partition regular. In other words, there is a partition of $\mathbb{N}$ into finitely many classes such that whenever $A\vec{x}=\vec{0}$ and $B\vec{y}=\vec{0}$ we cannot have all entries from $\vec{x}$ and $\vec{y}$ belonging to the same color class. It follows that there is no 'universal' kernel partition regular system, and in particular that (m,p,c)-systems are not universal for kernel partition regularity.

Given a matrix A, let us define a matrix A', representing the linear dependence of the rows of A, as follows. Let the rows of A be $\left\{\vec{r}_i : i\in I\right\}$. Choose a maximal linearly independent (over $\mathbb{Q}$) set of these rows: say $\left\{\vec{r}_j : j\in J\right\}$. Thus, for each $i\in I\setminus J$, we may write $\vec{r}_i$ as a rational linear combination of the $\vec{r}_j$: say

$$\vec{r}_i=\sum_{j\in J} q_j(i)\,\vec{r}_j,$$

where each $q_j\in\mathbb{Q}$ and only finitely many of the q_j are non-zero. We now let A' be the matrix whose rows are indexed by $I\setminus J$ and whose columns are indexed by I, with

$$A'_{ij}=\begin{cases} q_j(i) & \text{if } j\in J\\ -1 & \text{if } j=i\\ 0 & \text{otherwise.}\end{cases}$$

It is immediate that if A is partition regular then A' is kernel partition regular. Indeed, given a finite coloring of $\mathbb{N}$, choose an integer vector $\vec{x}$ with $A\vec{x}$ monochromatic, and set $y_i=(A\vec{x})_i$. Then by the definition of A' we have $A'\vec{y}=\vec{0}$.

In the converse direction, suppose that A' is kernel partition regular. Given a finite coloring of $\mathbb{N}$, choose a monochromatic vector $\vec{y}$ with $A'\vec{y}=0$. Then any vector $\vec{x}$ satisfying $(A\vec{x})_j=y_j$ for all $j\in J$ also satisfies $A\vec{x}=\vec{y}$. It follows that we may find a vector $\vec{x}$, with all its entries *rational*, such that $A\vec{x}$ is monochromatic

– indeed, to solve $(A\vec{x})_j = y_j$ for all $j \in J$ we are merely solving a family of linearly independent rational equations. So A is 'close' to being partition regular.

We are now ready for our main result.

Theorem 4.1. *There exist kernel partition regular matrices A and B, and a finite coloring of $\mathbb{N}$, such that if $A\vec{x} = \vec{0}$ and $B\vec{y} = \vec{0}$ then $\vec{x}$ and $\vec{y}$ cannot have all entries belonging to the same color class.*

Proof. Choose matrices A and B that generate the Milliken-Taylor systems for the sequences (1) and (1,2) respectively (these two sequences are just chosen for convenience). Then A and B are partition regular (by Theorem 2.5), and so the matrices A' and B' formed as above are kernel partition regular.

We claim, however, that the diagonal matrix formed from A' and B' is not kernel partition regular. Indeed, choose a coloring of $\mathbb{N}$ that separates A from B (this is possible by Theorem 3.3), and suppose that there are vectors $\vec{x}$ and $\vec{y}$, with all entries in the same class, such that $A'\vec{x} = 0$ and $B'\vec{y} = 0$. Then, by the above remarks, there are rational vectors $\vec{z}$ and $\vec{w}$ with all entries of $A\vec{z}$ and $B\vec{w}$ belonging to the same color class.

Now, each entry of $\vec{z}$ is also an entry of $A\vec{z}$, and so in fact each entry of $\vec{z}$ is an integer. Also, for any $i \geq 2$, both $w_{i-1} + w_i + 2w_{i+1}$ and $w_{i-1} + 2w_i + 2w_{i+1}$ are entries of $B\vec{w}$, and so w_i is an integer. However, this contradicts the choice of the coloring. ∎

We remark that it would not be very pleasant to write down explicitly the matrix B' in the above example.

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